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On Certain Integrals of Lipschitz-Hankel Type Involving Products of Bessel Functions

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ON CERTAIN INTEGRALS OF LIPSCHITZ–HANKEL TYPE INVOLVING PRODUCTS OF BESSEL FUNCTIONS

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CONTENTS

This paper is concerned with the evaluation and tabulation of certain integrals of the type

$$I(\mu, \nu; \lambda) = \int_0^\infty J_\mu(at)\, J_\nu(bt)\, \mathrm{e}^{-ct}\, t^\lambda \mathrm{d}t.$$

In part I of this paper, a formula is derived for the integrals in terms of an integral of a hypergeometric function. This new integral is evaluated in the particular cases which are of most frequent use in mathematical physics. By means of these results, approximate expansions are obtained for cases in which the ratio b/a is small or in which $b \simeq a$ and c/a is small.

In part II, recurrence relations are developed between integrals with integral values of the parameters μ, ν and λ. Tables are given by means of which $I(0, 0; 1)$, $I(0, 1; 1)$, $I(1, 0; 1)$, $I(1, 1; 1)$, $I(0, 0; 0)$, $I(1, 0; 0)$, $I(0, 1; 0)$, $I(1, 1; 0)$, $I(0, 1; -1)$, $I(1, 0; -1)$ and $I(1, 1; -1)$ may be evaluated for $0 \leqslant b/a \leqslant 2$, $0 \leqslant c/a \leqslant 2$.

1. Introduction

This paper is concerned with the evaluation and tabulation of certain integrals of the form

$$I(\mu, \nu; \lambda) = \int_0^\infty J_\mu(at)\, J_\nu(bt)\, \mathrm{e}^{-ct}\, t^\lambda\, \mathrm{d}t. \tag{1·1}$$

We shall also use the notation

$$J(\mu, \nu; \lambda) = \int_0^\infty J_\mu(\xi)\, J_\nu(\rho\xi)\, \mathrm{e}^{-\zeta\xi}\, \xi^\lambda\, \mathrm{d}\xi, \tag{1·2}$$

where we have written $\rho = b/a$, $\zeta = c/a$, $\xi = at$, so that

$$I(\mu, \nu; \lambda) = a^{-\lambda-1} J(\mu, \nu; \lambda). \tag{1·3}$$

These integrals are convergent if $c>0$ and $\mu+\nu+\lambda>-1$. If $c=0$ then we must have

$$\text{(i)}\quad \mu+\nu+1>-\lambda>-1 \quad \text{if} \quad a\neq b;$$

$$\text{(ii)}\quad \mu+\nu+1>-\lambda>0 \quad \text{if} \quad a=b.$$

In all cases considered it is assumed that μ, ν are real.

We shall show, in § 6 below, that when $a>b$ the integral (1·1) is equal to the integral

$$\frac{2}{\pi}\int_0^\infty K_\mu(ak)\,I_\nu(bk)\cos\{ck+\tfrac{1}{2}(\mu-\nu+\lambda)\,\pi\}\,k^\lambda\,\mathrm{d}k. \tag{1·4}$$

When $a<b$, interchange a, b and μ, ν. Convergence of this integral at the lower limit requires in general that $-\mu+\nu+\lambda>-1$.

It happens frequently that the solutions of problems in mathematical physics relating to systems with axial symmetry can be reduced to simple combinations of integrals of this type.

In most cases of practical interest the parameters μ, ν, λ are of the form m or $m+\frac{1}{2}$, where m is an integer. If either μ or ν or both are of the form $m+\frac{1}{2}$, the integrals which occur in physical applications are usually elementary. $J_{m+\frac{1}{2}}(z)$ can readily be expressed as a linear combination of terms of the form $z^{p+\frac{1}{2}}\cos z$, $z^{q+\frac{1}{2}}\sin z$, where p and q are integers, and the trigonometric terms can be absorbed into the exponential term in the integral (1·1). In this way the integrals can be reduced to a form involving only a single Bessel function. That these integrals can be evaluated easily is illustrated in Sneddon (1951, pp. 459–468). In the present paper we shall consider mainly the case in which μ, ν and λ are all integers.

As examples of the use of these integrals in physical problems we may quote the following:

(*a*) The potential in cylindrical co-ordinates (ρ, z, ϕ) due to a ring of radius a and carrying a charge Q is $QI(0,0;\,0)$ with $\rho=b$ and $z=c$ (Smythe 1939, p. 208).

(*b*) The vector potential due to a plane loop of wire of radius a carrying a current i has only one component given by $2\pi aiI(1,1;\,0)$ (Smythe 1939, p. 299).

(*c*) The stream function for a thin circular ring is proportional to $I(1,0;\,0)$ (Bateman 1944, p. 417).

(*d*) The potential due to a uniform distribution of charge over the surface of a disk is proportional to $I(1,0;\,-1)$ (Bateman 1944, pp. 410, 417).

(*e*) The components of stress in a semi-infinite solid (Terazawa 1916) and in a thick plate (Sneddon 1946) are proportional to several integrals of the type (1·1).

The problems listed above have obvious analogues in hydrodynamics and in the theory of steady-state conduction of heat, and the solutions of more complicated problems can often be obtained in the form of series of terms of the type (1·1).

It would appear that no systematic discussion or numerical evaluation of integrals of this type has been undertaken, though some special examples have been considered. For example, the special case in which $a=b$ has been discussed by Nomura (1941) who has provided a table of numerical values for a wide range of μ, ν and λ. Functions equivalent to $I(0,0;\,0)$ and $I(1,1;\,0)$ have been evaluated by Tallqvist (1932), though the tabulation is in terms of the parameters r and θ, where $\tan\theta=b/c$ and $r^2=1+b^2$, a being taken to be unity. The function $I(1,0;\,0)$ has been evaluated by Sura-Bura (1950); the expression found for this integral by Sura-Bura differs from that given below, but we show that the two forms are equivalent.

The remainder of this paper is divided into two parts. Part I deals with analytical transformation and evaluation of the integrals, and with series expansions for various values of the parameters. Part II deals with recurrence relations and numerical tables. The reader who wishes to use the tables need only consult part II.

PART I. EVALUATION OF THE INTEGRALS

2. Transformation of $I(\mu, \nu; \lambda)$

Graf's generalization of Neumann's addition theorem for Bessel functions (Watson 1944, p. 359) states that

$$J_\eta(R)\left(\frac{A-B\,\mathrm{e}^{-i\theta}}{A-B\,\mathrm{e}^{i\theta}}\right)^{\frac{1}{2}\eta} = \sum_{m=-\infty}^{\infty} J_{\eta+m}(A)\,J_m(B)\,\mathrm{e}^{im\theta}, \tag{2·1}$$

where

$$R^2 = A^2+B^2-2AB\cos\theta = (A-B\,\mathrm{e}^{-i\theta})\,(A-B\,\mathrm{e}^{i\theta}). \tag{2·2}$$

This formula is normally valid only if $|B\,\mathrm{e}^{\pm i\theta}| < |A|$, but when η is an integer this restriction may be removed.

If we multiply the numerator and denominator of the fraction on the left-hand side of equation (2·1) by $(A-B\,\mathrm{e}^{-i\theta})^{\frac{1}{2}\eta}$, multiply both sides of the equation by $\mathrm{e}^{-in\theta}$ and integrate with respect to θ from 0 to 2π we find that

$$J_{\eta+n}(A)\,J_n(B) = \frac{1}{\pi}\int_0^\pi \frac{J_\eta(R)}{R^\eta}(A-B\,\mathrm{e}^{-i\theta})^\eta\,\mathrm{e}^{-in\theta}\,\mathrm{d}\theta. \tag{2·3}$$

Substituting this expression in equation (1·1) with $A = at$, $B = bt$ and $\eta = \mu-\nu$, we obtain the formula

$$I(\mu,\nu;\lambda) = \frac{1}{\pi}\int_0^\pi (a-b\,\mathrm{e}^{-i\theta})^{\mu-\nu}\,\mathrm{e}^{-i\nu\theta}\,\mathrm{d}\theta\int_0^\infty r^{\nu-\mu}J_{\mu-\nu}(rt)\,e^{-ct}\,t^\lambda\,\mathrm{d}t, \tag{2·4}$$

in which we have written

$$r^2 = a^2+b^2-2ab\cos\theta. \tag{2·5}$$

Inserting the value for the inner integral given in Watson (1944, p. 384) we see that equation (2·4) reduces to

$$I(\mu,\nu;\lambda) = \frac{\Gamma(\mu-\nu+\lambda+1)}{2^{\mu-\nu}\pi c^{\mu-\nu+\lambda+1}\,\Gamma(\mu-\nu+1)}$$
$$\times\int_0^\pi (a-b\,\mathrm{e}^{-i\theta})^{\mu-\nu}\,\mathrm{e}^{-i\nu\theta}\,{}_2F_1\left(\tfrac{1}{2}\mu-\tfrac{1}{2}\nu+\tfrac{1}{2}\lambda+\tfrac{1}{2}, \tfrac{1}{2}\mu-\tfrac{1}{2}\nu+\tfrac{1}{2}\lambda+1; \mu-\nu+1; -\frac{r^2}{c^2}\right)\mathrm{d}\theta. \tag{2·6}$$

It should be emphasized that we have established this result for integral ν only. The inversion in the order of the integrations which must be effected to produce equation (2·6) is justified only if the inner integral is convergent, i.e. only if $\mu-\nu+\lambda > -1$. This will be true in all the cases considered below. Before proceeding to the evaluation of particular cases of the integral on the right-hand side of equation (2·6) we shall summarize the results on elliptic functions which we shall require later.

3. Elliptic integrals

Whenever we encounter an elliptic integral we shall employ the notation of Heuman (1941). The elliptic integrals of the first and second kinds are defined respectively by the equations

$$F(\alpha,\phi) = \int_0^\phi \frac{\mathrm{d}\psi}{\Delta(\psi)}, \quad E(\alpha,\phi) = \int_0^\phi \Delta(\psi)\,\mathrm{d}\psi, \tag{3·1}$$

in which
$$\Delta(\psi) = (1-k^2\sin^2\psi)^{\frac{1}{2}} \tag{3·2}$$
and
$$k = \sin\alpha. \tag{3·3}$$

When $\phi = \frac{1}{2}\pi$ these functions are denoted by $F(\alpha)$ and $E(\alpha)$ respectively. The functions tabulated by Heuman are
$$F_0(\alpha) = \frac{2}{\pi}F(\alpha), \quad E_0(\alpha) = \frac{2}{\pi}E(\alpha). \tag{3·4}$$

These complete elliptic integrals will also be denoted by $F_0(k)$, $F(k)$, etc.—or even F_0, F, etc.—in circumstances wherein there is no danger of confusion.

The following relations between the elliptic integrals of the first and second kinds are often useful in the evaluation of integrals:
$$\frac{dF}{dk} = \frac{E-k'^2F}{kk'^2}, \quad \frac{dE}{dk} = \frac{E-F}{k}, \tag{3·5}$$
where $k'^2 = 1-k^2$. For example, by their use it is readily established that
$$\int_0^{\frac{1}{2}\pi}\frac{d\psi}{\{\Delta(\psi)\}^3} = \frac{E(k)}{k'^2}, \quad \int_0^{\frac{1}{2}\pi}\frac{\sin^2\psi\, d\psi}{\{\Delta(\psi)\}^3} = \frac{E(k)-k'^2F(k)}{k^2k'^2}, \tag{3·6}$$
and that
$$\int_0^{\frac{1}{2}\pi}\frac{d\psi}{\{\Delta(\psi)\}^5} = \frac{1}{3(1-k^2)}\{4E-F\}. \tag{3·7}$$

Legendre's normal elliptic integral of the third kind is defined by
$$\Pi(\alpha,p,\phi) = \int_0^{\phi}\frac{d\psi}{(1-p\sin^2\psi)\,\Delta(\psi)}.$$

Suppose $k^2 \leqslant p \leqslant 1$ and introduce an angle β such that
$$\sin^2\beta = \frac{p-k^2}{pk'^2}, \quad \cos^2\beta = \frac{k^2(1-p)}{pk'^2}.$$

We now define $\Lambda(\alpha,\beta,\phi)$ by the relation
$$\Lambda(\alpha,\beta,\phi) = (1-p)^{\frac{1}{2}}(1-k^2/p)^{\frac{1}{2}}\,\Pi(\alpha,p,\phi), \tag{3·8}$$
and denote the corresponding complete integral $\Lambda(\alpha,\beta,\frac{1}{2}\pi)$ simply by $\Lambda(\alpha,\beta)$. The function tabulated by Heuman (1941) is
$$\Lambda_0(\alpha,\beta) = \frac{2}{\pi}\Lambda(\alpha,\beta),$$
which has derivatives
$$\frac{\partial\Lambda_0}{\partial\alpha} = -\frac{k'}{k}\frac{\sin\beta\cos\beta}{\Delta'(\beta)}(F_0-E_0), \quad \frac{\partial\Lambda_0}{\partial\beta} = \frac{E_0-k'^2\sin^2\beta F_0}{\Delta'(\beta)}, \tag{3·9}$$
with $\Delta'(\beta)$ denoting $\sqrt{(1-k'^2\sin^2\beta)}$.

A result which we shall require later is that
$$\int_0^{\frac{1}{2}\pi}\frac{(1-k^2\sin^2\psi)^{\frac{1}{2}}\,d\psi}{1-p\sin^2\psi} = \left\{\frac{p-k^2}{p(1-p)}\right\}^{\frac{1}{2}}\Lambda(\alpha,\beta)+\frac{k^2}{p}F(k). \tag{3·10}$$

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4. Evaluation of the integrals in particular cases

We shall now consider the evaluation of the integral $I(m,n;\,l)$ when the integers m, n, l take certain particular values.

Case (i): $I(n,n;\,0)$.

If we let $\mu=\nu=n$, $\lambda=0$ in equation (2·6), we find that the hypergeometric series occurring under the integral sign on the right-hand side reduces to ${}_2F_1(\frac{1}{2},1;\,1;\,-r^2/c^2)$, which is equal to $(1+r^2/c^2)^{-\frac{1}{2}}$. Substituting in (2·6) and writing

$$k^2=\frac{4ab}{(a+b)^2+c^2},$$

we find that

$$I(n,n;\,0)=\frac{(-1)^n k}{\pi\sqrt{(ab)}}\int_0^{\frac{1}{2}\pi}\frac{\cos(2n\psi)\,d\psi}{\sqrt{(1-k^2\sin^2\psi)}},$$

and, in particular, that

$$I(0,0;\,0)=\frac{k}{2\sqrt{(ab)}}F_0(k). \tag{4·1}$$

To evaluate $I(1,1;\,0)$ we put $\cos(2\psi)=-2k^{-2}\{(1-\frac{1}{2}k^2)-(1-k^2\sin^2\psi)\}$ and obtain the expression

$$I(1,1;\,0)=\frac{1}{k\sqrt{(ab)}}\{(1-\tfrac{1}{2}k^2)\,F_0(k)-E_0(k)\}. \tag{4·2}$$

Case (ii): $I(n,n;\,1)$

In this case $\mu=\nu=n$, $\lambda=1$ and the hypergeometric series occurring in the integral on the right takes the form ${}_2F_1(1,\frac{3}{2};\,1;\,-r^2/c^2)$, and this is equivalent to $(1+r^2/c^2)^{-\frac{3}{2}}$ so that we find that

$$I(n,n;\,1)=\frac{(-1)^n ck^3}{4\pi(ab)^{\frac{3}{2}}}\int_0^{\frac{1}{2}\pi}\frac{\cos(2n\psi)\,d\psi}{(1-k^2\sin^2\psi)^{\frac{3}{2}}}.$$

In particular we find that

$$I(0,0;\,1)=\frac{ck^3E_0(k)}{8k'^2(ab)^{\frac{3}{2}}}, \tag{4·3}$$

and that

$$I(1,1;\,1)=\frac{ck}{4(ab)^{\frac{3}{2}}}\{(1-\tfrac{1}{2}k^2)\,k'^{-2}\,E_0(k)-F_0(k)\}. \tag{4·4}$$

Case (iii): $I(n+1,n;\,-1)$

In cases in which μ does not equal ν the evaluation of the integral on the right-hand side of equation (2·6) is more complicated. If $\mu=n+1$, $\nu=n$, $\lambda=-1$ the hypergeometric function occurring in the integral reduces to

$${}_2F_1(\tfrac{1}{2},1;\,2;\,-r^2/c^2)=\frac{2c^2}{r^2}\left\{\left(1+\frac{r^2}{c^2}\right)^{\frac{1}{2}}-1\right\},$$

and we find that

$$I(n+1,n;\,-1)=I_1-I_2,$$

where

$$I_1=\frac{1}{\pi}\int_0^{\pi}(a-b\,e^{-i\theta})\,e^{-in\theta}\,r^{-2}(c^2+r^2)^{\frac{1}{2}}\,d\theta$$

and

$$I_2=\frac{c}{\pi}\int_0^{\pi}(a-b\,e^{-i\theta})\,e^{-in\theta}\,r^{-2}\,d\theta.$$

We shall consider the case $n=0$, i.e. the integral $I(1,0;\,-1)$. In this instance the evaluation of I_2 is elementary and gives

$$I_2=\begin{cases}0 & (a<b),\\ \dfrac{c}{2a} & (a=b),\\ \dfrac{c}{a} & (a>b).\end{cases} \tag{4·5}$$

On using (3·10), the expression for I_1 is found to be

$$\frac{2\surd(ab)}{\pi ka}E(k)+\frac{(a^2-b^2)\,k}{2\pi a(ab)^{\frac{1}{2}}}F(k)+\frac{a-b}{|\,a-b\,|}\frac{c}{\pi a}\Lambda(\alpha,\beta).$$

Finally we have

$$I(1,0;\,-1)=\begin{cases}\dfrac{(ab)^{\frac{1}{2}}}{ka}E_0(k)+\dfrac{(a^2-b^2)\,k}{4a(ab)^{\frac{1}{2}}}F_0(k)+\dfrac{c}{2a}\Lambda_0(\alpha,\beta)-\dfrac{c}{a} & (a>b),\\[2mm] \dfrac{E_0(k)}{k}-\dfrac{c}{2a} \quad (a=b), & \\[2mm] \dfrac{(ab)^{\frac{1}{2}}}{ka}E_0(k)+\dfrac{(a^2-b^2)\,k}{4a(ab)^{\frac{1}{2}}}F_0(k)-\dfrac{c}{2a}\Lambda_0(\alpha,\beta) & (a<b),\end{cases} \tag{4·6}$$

where

$$\sin^2\alpha=k^2,\quad \sin^2\beta=c^2\{(a-b)^2+c^2\}^{-1}. \tag{4·6a}$$

Case (iv): $I(n+1,n;\,0)$

With this choice of parameters the hypergeometric function in the integral on the right of equation (2·6) is

$${}_2F_1(1,\tfrac{3}{2};\,2;\,-r^2/c^2)=\frac{2c^2}{r^2}\left\{1-\frac{c}{\surd(c^2+r^2)}\right\}.$$

The case in which we are particularly interested has $n=0$. We obtain the formula

$$I(1,0;\,0)=\frac{1}{c}I_2-I_3,$$

where I_2 is the integral (4·5) above and

$$I_3=\frac{c}{\pi}\int_0^{\pi}\frac{(a-b\,\mathrm{e}^{-i\theta})}{r^2(r^2+c^2)^{\frac{1}{2}}}\,\mathrm{d}\theta$$

$$=\frac{kc}{4a\surd(ab)}F_0(k)+\frac{a-b}{|\,a-b\,|}\frac{1}{2a}\Lambda_0(\alpha,\beta).$$

We therefore find that

$$I(1,0;\,0)=\begin{cases}-\dfrac{kc}{4a\surd(ab)}F_0(k)-\dfrac{1}{2a}\Lambda_0(\alpha,\beta)+\dfrac{1}{a} & (a>b),\\[2mm] -\dfrac{kc}{4a^2}F_0(k)+\dfrac{1}{2a} & (a=b),\\[2mm] -\dfrac{kc}{4\,a\surd(ab)}F_0(k)+\dfrac{1}{2a}\Lambda_0(\alpha,\beta) & (a<b).\end{cases} \tag{4·7}$$

Case (v): $I(n+1,n;\,1)$

For this set of parameters the hypergeometric series becomes ${}_2F_1(1,\frac{3}{2};\,1;\,-r^2/c^2)$, which is equivalent to $(1+r^2/c^2)^{-\frac{3}{2}}$, so that we have

$$I(n+1,n;\,1)=\frac{1}{\pi}\int_0^{\pi}(a-b\,\mathrm{e}^{-in\theta})\,\mathrm{e}^{-i\theta}\,(c^2+r^2)^{-\frac{3}{2}}\,\mathrm{d}\theta,$$

from which it follows immediately that

$$I(1,0;\,1)=\frac{k^3(a^2-b^2-c^2)}{16ak'^2(ab)^{\frac{3}{2}}}E_0(k)+\frac{k}{4a\surd(ab)}F_0(k). \tag{4·8}$$

It follows from the differentiation formulae (3·5) that $I(n+1, n; l)$ for $l \geqslant 1$ can be expressed in terms of complete elliptic integrals of the first and second kinds only.

Case (vi): $I(1,1; -1)$

An integral which occurs frequently is $I(1,1; -1)$; it follows immediately from equations (4·2), (4·7) and (8·5) below, that

$$I(1,1;-1) = \begin{cases} \dfrac{cE_0}{2k\sqrt{(ab)}} - \dfrac{kc}{4ab\sqrt{(ab)}}(a^2+b^2+\tfrac{1}{2}c^2)F_0 + \dfrac{a^2-b^2}{4ab}\Lambda_0 + \dfrac{b}{2a} & (a>b), \\ \dfrac{c}{2ka}E_0 - \dfrac{kc}{4a^3}(2a^2+\tfrac{1}{2}c^2)F_0 + \tfrac{1}{2} & (a=b), \\ \dfrac{cE_0}{2k\sqrt{(ab)}} - \dfrac{kc}{4ab\sqrt{(ab)}}(a^2+b^2+\tfrac{1}{2}c^2)F_0 + \dfrac{b^2-a^2}{4ab}\Lambda_0 + \dfrac{a}{2b} & (a<b). \end{cases} \tag{4·9}$$

5. Approximate expressions for the integrals

In this section we shall derive certain expansions which are of use in calculating the integrals (1·1) when one or more of the parameters assumes small values. In accordance with the usual convention we shall denote integral values of λ, μ and ν by latin letters, say l, m and n.

Usually it is convenient to use form (1·2) and expand in terms of $\rho = b/a$, $\zeta = c/a$, but sometimes it is easier to use form (1·1) and expand directly in terms of a, b, c.

(a) *Expansions for small* ρ

If the parameter ρ is small we may replace $J_\nu(\xi\rho)$ in equation (1·2) by its series expansion. We then find that

$$J(m,n;l) = \frac{\rho^n}{2^n n!}\left\{j(m; l+n) - \frac{\rho^2}{4(n+1)}j(m; l+n+2) + \ldots\right\},$$

where

$$j(m;q) = \int_0^\infty J_m(\xi)\,\xi^q\,\mathrm{e}^{-\zeta\xi}\,\mathrm{d}\xi.$$

The integrals $j(m;q)$ can be evaluated in simple terms for a variety of values of m and q (Watson 1944, p. 386). We find in particular that

$$J(m,n;-n) = \frac{\{(1+\zeta^2)^{\frac{1}{2}}-\zeta\}^m}{2^n n!\sqrt{(1+\zeta^2)}}\rho^n + O(\rho^{n+2}),$$

$$J(m,n;-n-1) = \frac{\{(1+\zeta^2)^{\frac{1}{2}}-\zeta\}^m}{2^n n!\,m}\rho^n + O(\rho^{n+2}),$$

$$J(m,n;m-n) = \frac{1.3\ldots(2m-1)}{2^n n!\,(1+\zeta^2)^{m+\frac{1}{2}}}\rho^n + O(\rho^{n+2}),$$

$$J(m,n;m-n+1) = \frac{1.3\ldots(2m+1)\,\zeta}{2^n n!\,(1+\zeta^2)^{m+\frac{3}{2}}}\rho^n + O(\rho^{n+2}),$$

$$J(m,n;m-n+2) = \frac{1.3\ldots(2m+1)}{2^n n!\,(1+\zeta^2)^{m+\frac{5}{2}}}\rho^n + O(\rho^{n+2}).$$

These formulae are sufficient to deal with the integrals evaluated in this paper. Further terms and more general results can be obtained by using the relation

$$j(m;\, l+1) = -\frac{\partial}{\partial \zeta} j(m;\, l).$$

These expressions can be further simplified if $\zeta \ll 1$ or $\zeta \gg 1$. When $\zeta \ll 1$ we write (Watson 1944, p. 385)

$$j(m;\, l) = 2^l \left\{ \frac{\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2})}{\Gamma(\frac{1}{2}m-\frac{1}{2}l+\frac{1}{2})}\, {}_2F_1(\tfrac{1}{2}l+\tfrac{1}{2}m+\tfrac{1}{2}, \tfrac{1}{2}l-\tfrac{1}{2}m+\tfrac{1}{2};\, \tfrac{1}{2};\, -\zeta^2) \right.$$
$$\left. -2\zeta \frac{\Gamma(\frac{1}{2}l+\frac{1}{2}m+1)}{\Gamma(\frac{1}{2}m-\frac{1}{2}l)}\, {}_2F_1(\tfrac{1}{2}l+\tfrac{1}{2}m+1, \tfrac{1}{2}l-\tfrac{1}{2}m+1;\, \tfrac{3}{2};\, -\zeta^2) \right\}.$$

Using the expansions of the hypergeometric series, we can readily find $J(m, n;\, l)$ as an ascending series in ρ and ζ. The first terms in this expansion are

$$J(m, n;\, l) = \frac{2^l}{n!} \left\{ \frac{\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}l+\frac{1}{2})}{\Gamma(\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}l+\frac{1}{2})} - 2\zeta \frac{\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}l+1)}{\Gamma(\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}l)} \right\} \rho^n.$$

For $\zeta \gg 1$, it would be easy to expand $j(m;\, q)$ in terms of ζ^{-1}, but it is even simpler to proceed as follows: In order to take advantage of the symmetry it is convenient to proceed in terms of a, b and c instead of the parameters ρ and ζ. We write

$$I(m, n;\, l) = c^{-l-1} \int_0^\infty J_m(a\xi/c)\, J_n(b\xi/c)\, \xi^l\, \mathrm{e}^{-\xi}\, \mathrm{d}\xi;$$

then if $c \gg a$, $c \gg b$ it follows from the series expansions of J_m and J_n that

$$J_m(a\xi/c)\, J_n(b\xi/c) = \frac{a^m b^n}{(2c)^{m+n}\, \Gamma(m+1)\, \Gamma(n+1)} \left\{ \xi^{m+n} - \frac{1}{4c^2}\left(\frac{a^2}{m+1}+\frac{b^2}{n+1}\right) \xi^{m+n+2} + O(\xi^{m+n+4}) \right\},$$

and hence that

$$I(m, n;\, l) = \frac{a^m b^n}{2^{m+n} m!\, n!\, c^{l+m+n+1}} \left\{ (m+n+l)! - \frac{(m+n+l+2)!}{4c^2} \left(\frac{a^2}{m+1}+\frac{b^2}{n+1}\right) + O(a^4/c^4) \right\}.$$

(b) *Expansions for* $\rho \simeq 1$ *and* ζ *small*

When $\rho \simeq 1$ it is easiest to obtain the approximate values of the integrals from the explicit expressions for them in terms of elliptic integrals. If we put $\rho = 1+\gamma$, where γ is small but may be positive or negative, then we find that

$$k \simeq \rho^{\frac{1}{2}}\{1-\tfrac{1}{2}\gamma+\tfrac{1}{8}(2\gamma^2-\zeta^2)\}^{\frac{1}{2}}, \quad k' \simeq \tfrac{1}{2}(\gamma^2+\zeta^2)\,(1-\gamma).$$

The parameter k' is therefore small if $\rho \simeq 1$ and ζ is small. We may in these circumstances take

$$F_0(k) \simeq \frac{2}{\pi}\left\{\Lambda + \frac{(\Lambda-1)}{16}(\gamma^2+\zeta^2)\right\}, \quad E_0(k) \simeq \frac{2}{\pi}\left\{1 - \frac{(2\Lambda-1)}{16}(\gamma^2+\zeta^2)\right\}$$

(Jahnke & Emde 1945, p. 73), where $\Lambda = \ln(4/k')$;

$$\Lambda_0(\alpha, \beta) \simeq \beta E_0 - \tfrac{1}{4}(2F_0 - E_0)\, k'^2\, (\beta - \sin\beta\cos\beta)$$

(Heuman 1941, p. 134), where, from (4·6*a*),

$$\sin^2\beta = \frac{\zeta^2}{\gamma^2+\zeta^2}, \quad \cos^2\beta = \frac{\gamma^2}{\gamma^2+\zeta^2}, \quad \tan\beta = \left|\frac{\zeta}{\gamma}\right|.$$

It should be observed that in the way in which we have made use of the formula for $\Lambda_0(\alpha,\beta)$ in evaluating integrals, β always lies between 0 and $\frac{1}{2}\pi$ irrespective of the sign of γ. Hence $\tan\beta$ and $\cos\beta$ are positive, irrespective of the sign of γ. We obtain

$$\Lambda_0(\alpha,\beta) \simeq \frac{2}{\pi}\left\{\tan^{-1}\left|\frac{\zeta}{\gamma}\right| + \frac{\zeta|\gamma|}{16}(2\Lambda-1)(1-\zeta)\right\}.$$

On substituting these expansions in the expressions for the integrals in terms of elliptic integrals we obtain the following series:

$$J(1,0;1) = \frac{1}{2\pi(\gamma^2+\zeta^2)}\{-2\gamma+\Theta(\gamma^2+\zeta^2)-\zeta^2-\tfrac{3}{4}\gamma\Theta(\gamma^2+\zeta^2)+\tfrac{1}{8}\gamma(5\gamma^2+11\zeta^2)+\ldots\},$$

$$J(0,1;1) = \frac{1}{2\pi(\gamma^2+\zeta^2)}\{2\gamma+\Theta(\gamma^2+\zeta^2)-(2\gamma^2+\zeta^2)-\tfrac{5}{4}\gamma\Theta(\gamma^2+\zeta^2)+\tfrac{1}{8}\gamma(19\gamma^2+13\zeta^2)+\ldots\},$$

$$J(0,0;1) = \frac{\zeta}{\pi(\gamma^2+\zeta^2)}\{1-\tfrac{1}{2}\gamma+\tfrac{1}{8}\Theta(\gamma^2+\zeta^2)+\tfrac{3}{16}(\gamma^2-\zeta^2)+\ldots\},$$

$$J(1,1;1) = \frac{\zeta}{\pi(\gamma^2+\zeta^2)}\{1-\tfrac{1}{2}\gamma-\tfrac{3}{8}\Theta(\gamma^2+\zeta^2)+\tfrac{1}{16}(11\gamma^2+5\zeta^2)+\ldots\},$$

$$J(1,0;0) = \frac{1}{\pi}\tan^{-1}\left(\frac{\zeta}{\gamma}\right)-\frac{\zeta}{2\pi}\left\{\Theta-\frac{3\gamma\Theta}{4}+\tfrac{5}{8}\gamma+\tfrac{1}{16}\Theta(9\gamma^2-\zeta^2)-\tfrac{1}{16}(11\gamma^2-\zeta^2)+\ldots\right\},$$

$$J(0,1;0) = \frac{1}{1+\gamma}\left\{1-\frac{1}{\pi}\tan^{-1}\left(\frac{\zeta}{\gamma}\right)\right\}-\frac{\zeta}{2\pi}\{\Theta-\tfrac{5}{4}\gamma\Theta+\tfrac{3}{8}\gamma+\tfrac{1}{16}\Theta(21\gamma^2-\zeta^2)-\tfrac{1}{16}(9\gamma^2-\zeta^2)+\ldots\},$$

$$J(0,0;0) = \frac{1}{\pi}\{\Theta-\tfrac{1}{2}\gamma\Theta+\tfrac{1}{2}\gamma+\tfrac{1}{16}\Theta(5\gamma^2-\zeta^2)-\tfrac{1}{16}(7\gamma^2-\zeta^2)+\ldots\},$$

$$J(1,1;0) = \frac{1}{\pi}\{\Theta-2-\tfrac{1}{2}\Theta\gamma+\tfrac{3}{2}\gamma+\tfrac{3}{16}\Theta(3\gamma^2+\zeta^2)-\tfrac{1}{16}(21\gamma^2+\zeta^2)+\ldots\}.$$

It should be noted that:

(i) $\Lambda = \ln(4/k')$ has been replaced by $\Theta = \frac{1}{2}\ln\{64/(\gamma^2+\zeta^2)\}$. These two quantities are related by the expression $\Lambda \simeq \Theta+\frac{1}{2}\gamma-\frac{1}{8}(\gamma^2+\zeta^2)$.

(ii) When γ is negative, the negative value should be used directly in the formulae, with the proviso that $\tan^{-1}(\zeta/\gamma)$ is replaced by $\pi-\tan^{-1}(-\zeta/\gamma)$ *not* $-\tan^{-1}(-\zeta/\gamma)$.

(iii) Some of the integrals have singularities at $\rho = 1$, $\zeta = 0$. We include special tables for the integrals in this critical region, but these values were not computed from the above series expansions. The series expansions have been used to indicate the nature of the singularities. When these are known, instead of computing the integral itself the singularity has been removed by some simple device and the resulting relatively smooth function computed directly from the explicit expressions for the integrals in terms of elliptic integrals.

(c) *Expansions for small values of c*

The expansion of $I(m,n;l)$ as a power series in c may be found from equation (2·6) by expressing the hypergeometric function with argument $(-r^2/c^2)$ as the sum of two hypergeometric functions with arguments $(-c^2/r^2)$ and then expanding both functions as power series in $(-c^2/r^2)$. Using a well-known relation for hypergeometric series (Magnus & Oberhettinger 1949, p. 9) we find that

$$I(m,n;\,l) = \frac{2^l\Gamma\left(\frac{m-n+l+1}{2}\right)}{\pi\Gamma\left(\frac{m-n-l+1}{2}\right)}\int_0^\pi (a-b\,\mathrm{e}^{-i\theta})^{m-n}\,\mathrm{e}^{-in\theta}\,r^{-m+n-l-1}\,{}_2F_1\left(\frac{m-n+l+1}{2},\frac{-m+n+l+1}{2};\frac{1}{2};-\frac{c^2}{r^2}\right)$$

$$-\frac{2^{l+1}\Gamma\left(\frac{m-n+l+2}{2}\right)}{\pi\Gamma\left(\frac{m-n-l}{2}\right)}\,c\int_0^\pi (a-b\,\mathrm{e}^{-i\theta})^{m-n}\,\mathrm{e}^{-in\theta}\,r^{-m+n-l-2}\,{}_2F_1\left(\frac{m-n+l+2}{2},\frac{-m+n+l+2}{2};\frac{3}{2};-\frac{c^2}{r^2}\right)$$

If we use the abbreviations

$$G(d,s) = \frac{2(-1)^s}{\pi(a+b)^d}\int_0^{\frac{1}{2}\pi}\frac{\cos(2s\phi)\,\mathrm{d}\phi}{(1-h^2\sin^2\phi)^{\frac{1}{2}d}},\quad h^2 = \frac{4ab}{(a+b)^2},$$

$$H(n) = \frac{1}{\pi}\int_0^\pi (a-b\,\mathrm{e}^{-i\theta})\,\mathrm{e}^{-in\theta}\frac{\mathrm{d}\theta}{r^2},$$

we find on expanding the hypergeometric series in powers of $(-r^2/c^2)$ that

$$I(n,n;\,0) = G(1,n)-\tfrac{1}{2}c^2G(3,n)+\ldots,$$

$$I(n,n;\,1) = c\{G(3,n)-\tfrac{3}{2}c^2G(s,n)+\ldots\},$$

$$I(n+1,n;\,-1) = -cH(n)+\{aG(1,n)-bG(1,n+1)\}+\tfrac{1}{2}c^2\{aG(3,n)-bG(3,n+1)\}+\ldots,$$

$$I(n+1,n;\,0) = H(n)-c\{aG(3,n)-bG(3,n+1)\}+\tfrac{1}{2}c^3\{aG(5,n)-bG(5,n+1)\}+\ldots,$$

$$I(n+1,n;\,1) = \{aG(3,n)-bG(3,n+1)\}-\tfrac{3}{2}c^2\{aG(5,n)-bG(5,n+1)\}+\ldots.$$

The particular cases in which we are most closely interested can then be readily derived from the formulae

$$G(1,0) = \frac{1}{a+b}F_0(h),\qquad G(1,1) = \frac{a+b}{2ab}\left\{\frac{a^2+b^2}{(a+b)^2}F_0(h)-E_0(h)\right\},$$

$$G(3,0) = \frac{E_0(h)}{(a+b)\,(a-b)^2},\qquad G(3,1) = \frac{1}{2ab(a+b)}\left\{\frac{a^2+b^2}{(a-b)^2}E_0(h)-F_0(h)\right\},$$

$$G(5,0) = \frac{1}{3(a+b)^3\,(a-b)^2}\left\{\frac{4(a^2+b^2)}{(a-b)^2}E_0-F_0\right\},$$

$$G(5,1) = \frac{1}{6ab(a+b)\,(a-b)^2}\left\{\frac{a^4+14a^2b^2+b^4}{(a^2-b^2)^2}E_0-F_0\right\},$$

$$H(0) = \begin{cases} 0 & (a<b),\\ 1/(2a) & (a=b),\\ 1/a & (a>b).\end{cases}$$

It should be noted that the elliptic functions may be expressed in terms of b/a instead of h, by means of a Landen transformation.

6. Equivalence of the two forms of integral (1·1) and (1·4)

In order to prove the equivalence of the integrals (1·1) and (1·4) we make use of the following notation for Bessel functions (Jeffreys 1946):

$$Kh_\nu(z) = \frac{2}{\pi} K_\nu(z), \tag{6·1}$$

$$Hs_\nu(z) = J_\nu(z) + \mathrm{i}Y_\nu(z) = H_\nu^{(1)}(z), \tag{6·2}$$

$$Hi_\nu(z) = J_\nu(z) - \mathrm{i}Y_\nu(z) = H_\nu^{(2)}(z). \tag{6·3}$$

The advantage of the Hs, Hi notation is that Hs tends to zero as z tends to infinity in the upper (superior) half of the complex z-plane and Hi in the lower (inferior) half-plane. This is indicated by the notation. We shall require the results that, if x is real,

$$J_\nu(\mathrm{i}x) = \mathrm{e}^{\frac{1}{2}\pi\nu\mathrm{i}} I_\nu(x), \quad J_\nu(-\mathrm{i}x) = \mathrm{e}^{-\frac{1}{2}\pi\nu\mathrm{i}} I_\nu(x); \tag{6·4}$$

these results are immediate consequences of the expressions for $J_\nu(z)$ and $I_\nu(z)$ as infinite series. In addition, we need the relations

$$Hs_\nu(z) = \frac{J_{-\nu}(z) - \mathrm{e}^{-\nu\pi\mathrm{i}} J_\nu(z)}{\mathrm{i}\sin\nu\pi}, \quad Hi_\nu(z) = \frac{J_{-\nu}(z) - \mathrm{e}^{\nu\pi\mathrm{i}} J_\nu(z)}{-\mathrm{i}\sin\nu\pi}, \tag{6·5a}$$

and

$$Kh_\nu(z) = \frac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu\pi}, \tag{6·5b}$$

which are valid when ν is not an integer. When ν is an integer we replace the right-hand sides of these equations by their limits (Watson 1944, pp. 73 and 78).

From these relations we can readily show that

$$Hs_\nu(\mathrm{i}x) = -\mathrm{i}\,\mathrm{e}^{-\frac{1}{2}\nu\pi\mathrm{i}} Kh_\nu(x), \quad Hs_\nu(-\mathrm{i}x) = 2\,\mathrm{e}^{-\frac{1}{2}\pi\nu\mathrm{i}} I_\nu(x) - \mathrm{i}\,\mathrm{e}^{\frac{1}{2}\pi\nu\mathrm{i}} Kh_\nu(x), \tag{6·6}$$

and that

$$Hi_\nu(\mathrm{i}x) = 2\mathrm{e}^{\frac{1}{2}\nu\pi\mathrm{i}} I_\nu(x) + \mathrm{i}\,\mathrm{e}^{-\frac{1}{2}\nu\pi\mathrm{i}} Kh_\nu(x), \quad Hi_\nu(-\mathrm{i}x) = \mathrm{i}\,\mathrm{e}^{\frac{1}{2}\pi\nu\mathrm{i}} Kh_\nu(x). \tag{6·7}$$

In this notation we may write

$$I(\mu,\nu;\lambda) = \frac{1}{4}\int_0^\infty Hs_\mu(at)\,Hi_\nu(bt)\,\mathrm{e}^{-ct}t^\lambda\,\mathrm{d}t + \frac{1}{4}\int_0^\infty Hs_\mu(at)\,Hs_\nu(bt)\,\mathrm{e}^{-ct}t^\lambda\,\mathrm{d}t$$
$$+\frac{1}{4}\int_0^\infty Hi_\mu(at)\,Hi_\nu(bt)\,\mathrm{e}^{-ct}t^\lambda\,\mathrm{d}t + \frac{1}{4}\int_0^\infty Hi_\mu(at)\,Hs_\nu(bt)\,\mathrm{e}^{-ct}t^\lambda\,\mathrm{d}t. \tag{6·8}$$

Now if we integrate the function $\frac{1}{4}Hs_\mu(a\tau)\,Hi_\nu(b\tau)\,\mathrm{e}^{-c\tau}\tau^\lambda$ round a contour C_1 in the upper right-hand quadrant of the complex τ-plane consisting of the positive real axis from $(0,0)$ to $(0,R)$, a quarter circle $R\mathrm{e}^{\mathrm{i}\theta}$ $(0 \leqslant \theta \leqslant \frac{1}{2}\pi)$ and the positive imaginary axis $(\mathrm{i}R,0)$ to $(0,0)$

$$\int_{C_1} \mathrm{H}s_\mu(a\tau)\,Hi_\nu(b\tau)\,\mathrm{e}^{-c\tau}\tau^\lambda\,\mathrm{d}\tau = 0.$$

From the asymptotic expansions of the functions $Hs_\mu(a\tau)$, $Hi_\nu(b\tau)$ we can readily show that if $a > b$ the integral along the circular arc tends to the value zero as $R \to \infty$. We therefore have that the integral of the function along the real axis from 0 to ∞ is equal to its integral along the imaginary axis from 0 to $i\infty$. Transforming this latter integral into an integration along the real axis by means of the relations (6·6) and (6·7) we find that

$$\frac{1}{4}\int_0^\infty Hs_\mu(at)\, Hi_\nu(bt)\, e^{-ct}\, t^\lambda\, dt$$

$$= \frac{1}{2}\int_0^\infty Kh_\nu(bt)\, I_\mu(at)\, t^\lambda\, e^{-ict-\frac{1}{2}\mu\pi i+\frac{1}{2}\nu\pi i-\frac{1}{2}\lambda\pi i}\, dt$$

$$+\frac{1}{4}\int_0^\infty Kh_\mu(at)\, Kh_\nu(bt)\, t^\lambda\, e^{-ict+\frac{1}{2}\mu\pi i+\frac{1}{2}\nu\pi i-\frac{1}{2}(\lambda+1)\pi i}\, dt.$$

In this way we transform the first term of the sum (6·8). The second term may be dealt with similarly by integrating the function $\frac{1}{4}Hs_\mu(a\tau)\, Hs_\nu(b\tau)\, e^{-c\tau}\, \tau^\lambda$ along the same contour C_1. The third and fourth terms are transformed in the same kind of way by integrating the functions $\frac{1}{4}Hi_\mu(a\tau)\, Hi_\nu(b\tau)\, e^{-c\tau}\, \tau^\lambda$ and $\frac{1}{4}Hi_\mu(a\tau)\, Hs_\nu(b\tau)\, e^{-c\tau}\, \tau^\lambda$ along a contour C_2 in the lower right-hand quadrant in the complex τ-plane. Transforming the four terms of the right-hand side of equation (6·8) in this way and adding we obtain the equation (1·4). It will be observed that this result is valid only if $a>b$. If $a<b$, the roles of the contours C_1 and C_2 are interchanged.

Appendix. An alternative expression for $I(1, 0; 0)$

The object of this appendix is to show that our expression (4·7) for the integral $I(1, 0; 0)$ may be transformed to yield a form previously derived by Sura-Bura (1950). In order to establish the equivalence of our formula to Sura-Bura's we first of all develop what would appear to be a new type of transformation for elliptical integrals of the third kind, based on the well-known transformation of Landen for elliptic integrals of the first and second kinds.

The transformation is as follows:

Suppose that two elliptic integrals have parameters k, β and k_1, β_1, respectively, related by the equations

$$k_1 = \frac{1-k'}{1+k'}, \quad \sin\beta_1 = \frac{(1+k')\sin\beta\cos\beta}{\sqrt{(1-k^2\sin^2\beta)}}, \tag{A. 1}$$

where $k'^2 = 1-k^2$. Then it is well known that

$$F(k,\beta) = \frac{F(k_1,\beta_1)}{1+k'}, \quad E(k,\beta) = \tfrac{1}{2}(1+k')\, E(k_1,\beta_1) - \frac{k'}{1+k'}F(k_1,\beta_1) + \tfrac{1}{2}(1-k')\sin\beta_1, \tag{A. 2}$$

and in particular that

$$F(k) = \frac{2}{1+k'}F(k_1), \quad E(k) = (1+k')\, E(k_1) - \frac{2k'}{1+k'}F(k_1). \tag{A. 3}$$

Furthermore, by Heuman (1941, p. 133) we have

$$\Lambda_0(k,\beta) = F_0(k)\, E(k',\beta) - \{F_0(k) - E_0(k)\}\, F(k',\beta). \tag{A. 4}$$

Replacing the complete elliptic integrals in equation (A. 4) by the corresponding formulae given by equations (A. 3) we find that

$$\Lambda_0(k,\beta) = \frac{2}{1+k'}F_0(k_1)\, E(k',\beta) - \{2F_0(k_1) - (1+k')\, E_0(k_1)\}\, F(k',\beta). \tag{A. 5}$$

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Now equations (A. 1) are equivalent to

$$k' = \frac{1-k_1}{1+k_1}, \quad \sin\beta = \frac{(1+k_1)\sin\beta_1\cos\beta_1}{\surd(1-k_1^2\sin^2\beta_1)}, \tag{A. 6}$$

so that writing down the value of $\Lambda_0(k_1,\beta_1)$ given by equation (A. 4) and replacing the incomplete integrals by equations (A. 2) we find that

$$\Lambda_0(k_1,\beta_1) = \tfrac{1}{2}(1+k_1)\,F_0(k_1)\,E(k',\beta)+\tfrac{1}{2}(1-k_1)\sin\beta F_0(k_1)-\left\{F_0(k_1)-\frac{1}{1+k_1}E_0(k_1)\right\}F(k',\beta). \tag{A. 7}$$

Comparing equations (A. 5) and (A. 7) and remembering that

$$1+k_1 = \frac{2}{1+k'},$$

we obtain the relation

$$\tfrac{1}{2}\Lambda_0(k,\beta) = \Lambda_0(k_1,\beta_1)-\tfrac{1}{2}(1-k_1)\,F_0(k_1)\sin\beta. \tag{A. 8}$$

The formula given by Sura-Bura (1950) is

$$I(1,0;\,0) = \frac{1}{a}\{1-\Lambda_0(k_1,\beta_1)\}, \tag{A. 9a}$$

where

$$k_1^2 = \frac{x_2(x_1-2c^2)}{x_1(x_2+2a^2)}, \quad \sin^2\beta_1 = \frac{x_1}{x_1+x_2} \tag{A. 9b}$$

with

$$x_1 = W-a^2+b^2+c^2, \quad x_2 = W+a^2-b^2-c^2 \tag{A. 9c}$$

and

$$W^2 = (a^2+b^2+c^2)^2-4a^2b^2. \tag{A. 9d}$$

We shall now prove that the formula contained in the equations (A. 9) is equivalent to our equation (4·7).

It is a matter of simple manipulation to show that the equations (A. 9b) can be written in the form

$$k_1^2 = \left(\frac{1-k'}{1+k'}\right)^2, \quad \sin\beta_1 = \frac{(1+k')\sin\beta\cos\beta}{\surd(1-k^2\sin^2\beta)}, \tag{A. 10}$$

where

$$\sin^2\beta = \frac{c^2}{(a-b)^2+c^2} \tag{A. 11}$$

and

$$k'^2 = \frac{(a-b)^2+c^2}{(a+b)^2+c^2}, \quad k^2 = \frac{4ab}{(a+b)^2+c^2}. \tag{A. 12}$$

Hence the $\Lambda_0(k_1,\beta_1)$ which occurs in Sura-Bura's formula (A. 9a) may be replaced by the expression

$$\Lambda_0(k_1,\beta_1) = \tfrac{1}{2}\Lambda_0(k,\beta)+\frac{kc}{4\surd(ab)}F_0(k). \tag{A. 13}$$

There is one further point to be observed. The second of equations (A. 10) for $\sin\beta_1$ can be written in the form

$$\sin\beta = \frac{(1+k_1)\sin(2\beta_1)}{2\surd(1-k_1'^2\sin^2\beta_1)}, \tag{A. 14}$$

where β_1 is given by the second of equations (A. 9b) and therefore lies in the interval $(0,\tfrac{1}{2}\pi)$. From equations (A. 9c) we know that $x_1 > x_2$ if $a < b$ and that $x_1 < x_2$ if $a > b$. Hence from the second of equations (A. 9b), $\tfrac{1}{4}\pi < \beta_1 < \tfrac{1}{2}\pi$ if $a < b$ and $0 < \beta_1 < \tfrac{1}{4}\pi$ if $a > b$. From equation

(A. 14) β varies from 0 to π as β_1 varies from 0 to $\frac{1}{2}\pi$, but it is not clear whether $\beta_1 = 0$ should correspond to $\beta = 0$ and $\beta_1 = \frac{1}{2}\pi$ to $\beta = \pi$, or whether $\beta_1 = 0$ should correspond to $\beta = \pi$ and $\beta_1 = \frac{1}{2}\pi$ to $\beta = 0$. To decide this point we consider the limiting form of equation (A. 13) when $a = 0$ and make use of the well-known result (Heuman 1941, p. 135)

$$\Lambda_0(k, \pi - \beta) = 2 - \Lambda_0(k, \beta) \quad (0 < \beta < \tfrac{1}{2}\pi), \tag{A. 15}$$

a special form of which is

$$\Lambda_0(0, \beta) = \begin{cases} \sin\beta & (0 < \beta < \frac{1}{2}\pi), \\ 1 & (\beta = \frac{1}{2}\pi), \\ 2 - \sin\beta & (\frac{1}{2}\pi < \beta < \pi). \end{cases} \tag{A. 16}$$

When $a = 0$, $x_1 = 2(b^2 + c^2)$, $x_2 = 0$, $k_1 = 0$, $\beta_1 = \frac{1}{2}\pi$, $k = 0$, $\sin^2\beta = c^2/(b^2 + c^2)$. From (A. 13) we therefore have

$$\Lambda_0(0, \tfrac{1}{2}\pi) = \tfrac{1}{2}\Lambda_0(0, \beta) + \tfrac{1}{2}\sin\beta,$$

and we see from the relations (A. 16) that this indicates that β lies in the interval $(\frac{1}{2}\pi, \pi)$. Hence when $a < b$, $\frac{1}{2}\pi < \beta < \pi$. If therefore β^* is the value of β in the range $0 < \beta < \frac{1}{2}\pi$ which satisfies equation (A. 11) then

$$\Lambda_0(k_1, \beta_1) = \begin{cases} 1 - \frac{1}{2}\Lambda_0(k, \beta^*) + \dfrac{kc}{4\sqrt{(ab)}} F_0(k) & (a < b); \\[2ex] \frac{1}{2}\Lambda_0(k, \beta^*) + \dfrac{kc}{4\sqrt{(ab)}} F_0(k) & (a > b). \end{cases}$$

If we substitute this result in Sura-Bura's formula (A. 9a) we obtain our result (4·7).

PART II. TABLES OF INTEGRALS

7. Numerical values of the integrals

In part II we give numerical tables of the integral

$$I(\mu, \nu; \lambda) = \int_0^\infty J_\mu(at)\, J_\nu(bt)\, e^{-ct}\, t^\lambda\, dt,$$

and some information of recurrence relations which will be useful to the user of the tables.

The formulae derived in §4 and the recurrence relations in §9 below have been used to compute the values of the integrals $I(0,0;1)$, $I(0,1;1)$, $I(1,0;1)$, $I(1,1;1)$, $I(0,0;0)$, $I(1,0;0)$, $I(0,1;0)$, $I(1,1;0)$, $I(0,1;-1)$, $I(1,0;-1)$ and $I(1,1;-1)$ for values of a, b, c in the range $0 \leqslant b/a$, $c/a \leqslant 2$, with isolated values at $b/a = 3$, $c/a = 3$. The results of these calculations are shown in tables 1 to 11.

To facilitate interpolation for values of a, b, c intermediate to those listed in tables 1 to 11 a set of auxiliary tables is provided. For example, the integral $I(0,0;1)$ has a singularity at the point $a = b$, $c = 0$ so that interpolation in table 1 in the neighbourhood of that point would be an unsatisfactory procedure. For that reason the function

$$\frac{a}{c}\{(a-b)^2 + c^2\} \int_0^\infty J_0(at)\, J_0(bt)\, e^{-ct}\, t\, dt$$

is tabulated in table 1 (a) for a range of values of $\rho = b/a$ and $\zeta = c/a$ in the neighbourhood of the singularity. Tables 2 (a) to 8 (a) supplement tables 2 to 8 respectively in a similar way.

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Table 1. Values of the integral $a^2\int_0^\infty J_0(at)\,J_0(bt)\,e^{-ct}\,t\,dt$ for ranges of values of $\rho=b/a$ and $\zeta=c/a$

ρ \ ζ	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0	3·0
0·0	0·0000	0·1886	0·3202	0·3783	0·3809	0·3536	0·3148	0·2749	0·2382	0·2062	0·1789	0·0949
0·2	0·0000	0·2047	0·3399	0·3924	0·3881	0·3561	0·3150	0·2740	0·2370	0·2050	0·1778	0·0944
0·4	0·0000	0·2653	0·4063	0·4346	0·4075	0·3618	0·3143	0·2708	0·2331	0·2013	0·1746	0·0932
0·6	0·0000	0·4285	0·5399	0·5001	0·4310	0·3656	0·3100	0·2640	0·2261	0·1949	0·1691	0·0911
0·8	0·0000	0·9083	0·7380	0·5605	0·4416	0·3591	0·2986	0·2521	0·2154	0·1858	0·1616	0·0883
1·0	∞	1·6089	0·8185	0·5540	0·4191	0·3355	0·2776	0·2344	0·2009	0·1740	0·1520	0·0849
1·2	0·0000	0·7380	0·5982	0·4545	0·3596	0·2947	0·2476	0·2116	0·1831	0·1600	0·1408	0·0809
1·4	0·0000	0·2762	0·3481	0·3243	0·2830	0·2446	0·2123	0·1856	0·1633	0·1445	0·1286	0·0765
1·6	0·0000	0·1305	0·2017	0·2198	0·2120	0·1952	0·1766	0·1589	0·1428	0·1285	0·1158	0·0718
1·8	0·0000	0·0729	0·1243	0·1497	0·1563	0·1526	0·1440	0·1337	0·1230	0·1128	0·1032	0·0669
2·0	0·0000	0·0455	0·0817	0·1046	0·1158	0·1186	0·1163	0·1113	0·1049	0·0980	0·0911	0·0620
3·0	0·0000	0·0095	0·0185	0·0265	0·0331	0·0382	0·0420	0·0445	0·0459	0·0464	0·0462	0·0400

Table 2. Values of the integral $a^2\int_0^\infty J_0(at)\,J_1(bt)\,e^{-ct}\,t\,dt$ for ranges of values of $\rho=b/a$ and $\zeta=c/a$

ρ \ ζ	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0	3·0
0·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
0·2	−0·1047	−0·0860	−0·0463	−0·0112	0·0096	0·0185	0·0206	0·0195	0·0173	0·0148	0·0125	0·0054
0·4	−0·2432	−0·1888	−0·0864	−0·0101	0·0283	0·0420	0·0434	0·0398	0·0346	0·0294	0·0247	0·0106
0·6	−0·4933	−0·3249	−0·0950	0·0226	0·0649	0·0737	0·0694	0·0609	0·0519	0·0436	0·0364	0·0156
0·8	−1·2335	−0·4199	0·0111	0·1099	0·1231	0·1131	0·0975	0·0820	0·0683	0·0568	0·0472	0·0202
1·0	$-\infty$ $+\infty$	0·4252	0·3100	0·2406	0·1911	0·1536	0·1243	0·1012	0·0829	0·0683	0·0567	0·0244
1·2	1·8013	1·0294	0·5323	0·3396	0·2432	0·1848	0·1451	0·1163	0·0945	0·0776	0·0644	0·0281
1·4	0·8923	0·7453	0·5169	0·3598	0·2627	0·2000	0·1569	0·1257	0·1023	0·0843	0·0701	0·0312
1·6	0·5729	0·5258	0·4258	0·3292	0·2544	0·1997	0·1594	0·1292	0·1061	0·0881	0·0737	0·0337
1·8	0·4097	0·3896	0·3408	0·2838	0·2317	0·1888	0·1547	0·1277	0·1064	0·0892	0·0754	0·0356
2·0	0·3114	0·3013	0·2747	0·2402	0·2049	0·1729	0·1455	0·1227	0·1038	0·0883	0·0754	0·0369
3·0	0·1215	0·1203	0·1170	0·1120	0·1055	0·0982	0·0906	0·0828	0·0753	0·0682	0·0617	0·0368

Table 3. Values of the integral $a^2\int_0^\infty J_1(at)\,J_0(bt)\,e^{-ct}\,t\,dt$ for ranges of values of $\rho=b/a$ and $\zeta=c/a$

ρ \ ζ	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0	3·0
0·0	1·0000	0·9429	0·8004	0·6305	0·4761	0·3536	0·2624	0·1964	0·1489	0·1145	0·0894	0·0316
0·2	1·0312	0·9653	0·8065	0·6255	0·4676	0·3455	0·2561	0·1918	0·1456	0·1123	0·0878	0·0313
0·4	1·1412	1·0386	0·8187	0·6044	0·4391	0·3207	0·2373	0·1784	0·1362	0·1057	0·0832	0·0303
0·6	1·4106	1·1761	0·8068	0·5470	0·3837	0·2782	0·2070	0·1572	0·1216	0·0954	0·0760	0·0288
0·8	2·2570	1·2991	0·6731	0·4242	0·2972	0·2198	0·1676	0·1304	0·1031	0·0826	0·0669	0·0268
1·0	$+\infty$ $-\infty$	0·4252	0·3100	0·2406	0·1911	0·1536	0·1243	0·1012	0·0829	0·0683	0·0567	0·0244
1·2	−1·0648	−0·3882	−0·0193	0·0742	0·0941	0·0921	0·0833	0·0730	0·0631	0·0541	0·0463	0·0218
1·4	−0·4021	−0·2822	−0·1118	−0·0151	0·0282	0·0452	0·0498	0·0487	0·0453	0·0409	0·0364	0·0192
1·6	−0·2119	−0·1764	−0·1054	−0·0448	−0·0063	0·0150	0·0255	0·0298	0·0306	0·0295	0·0276	0·0165
1·8	−0·1294	−0·1155	−0·0829	−0·0481	−0·0206	−0·0019	0·0096	0·0161	0·0192	0·0203	0·0202	0·0140
2·0	−0·0862	−0·0796	−0·0631	−0·0429	−0·0245	−0·0100	0·0002	0·0069	0·0110	0·0132	0·0142	0·0117
3·0	−0·0211	−0·0206	−0·0191	−0·0168	−0·0141	−0·0112	−0·0083	−0·0057	−0·0034	−0·0014	0·0001	0·0037

Table 4. Values of the integral $a^2\int_0^\infty J_1(at)\,J_1(bt)\,\mathrm{e}^{-ct}\,t\,\mathrm{d}t$ for ranges of values of $\rho=b/a$ and $\zeta=c/a$

ρ \ ζ	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0	3·0
0·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
0·2	0·0000	0·0581	0·0865	0·0852	0·0699	0·0527	0·0383	0·0275	0·0198	0·0144	0·0106	0·0028
0·4	0·0000	0·1439	0·1971	0·1800	0·1404	0·1028	0·0736	0·0526	0·0379	0·0276	0·0204	0·0055
0·6	0·0000	0·3221	0·3619	0·2865	0·2067	0·1457	0·1027	0·0731	0·0527	0·0385	0·0286	0·0079
0·8	0·0000	0·8106	0·5860	0·3825	0·2548	0·1742	0·1218	0·0868	0·0630	0·0465	0·0348	0·0100
1·0	∞	1·5239	0·6945	0·4096	0·2662	0·1820	0·1284	0·0927	0·0682	0·0510	0·0387	0·0116
1·2	0·0000	0·6796	0·5049	0·3416	0·2369	0·1688	0·1230	0·0911	0·0685	0·0522	0·0403	0·0128
1·4	0·0000	0·2376	0·2806	0·2381	0·1860	0·1425	0·1090	0·0838	0·0649	0·0506	0·0398	0·0135
1·6	0·0000	0·1036	0·1524	0·1542	0·1356	0·1126	0·0913	0·0733	0·0587	0·0471	0·0379	0·0138
1·8	0·0000	0·0533	0·0875	0·0992	0·0960	0·0858	0·0736	0·0617	0·0512	0·0423	0·0349	0·0138
2·0	0·0000	0·0307	0·0534	0·0652	0·0677	0·0643	0·0581	0·0508	0·0437	0·0371	0·0314	0·0135
3·0	0·0000	0·0046	0·0087	0·0122	0·0148	0·0165	0·0173	0·0175	0·0171	0·0163	0·0153	0·0096

Table 5. Values of the integral $a\int_0^\infty J_0(at)\,J_0(bt)\,\mathrm{e}^{-ct}\,\mathrm{d}t$ for ranges of values of $\rho=b/a$ and $\zeta=c/a$

ρ \ ζ	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0	3·0
0·0	1·0000	0·9806	0·9285	0·8575	0·7809	0·7071	0·6402	0·5812	0·5300	0·4856	0·4472	0·3162
0·2	1·0102	0·9891	0·9331	0·8587	0·7800	0·7053	0·6381	0·5793	0·5283	0·4842	0·4460	0·3157
0·4	1·0441	1·0161	0·9466	0·8612	0·7764	0·6994	0·6318	0·5734	0·5231	0·4797	0·4422	0·3141
0·6	1·1146	1·0669	0·9657	0·8607	0·7675	0·6880	0·6206	0·5633	0·5144	0·4724	0·4361	0·3115
0·8	1·2702	1·1449	0·9772	0·8484	0·7490	0·6694	0·6039	0·5490	0·5024	0·4624	0·4277	0·3079
1·0	∞	1·1721	0·9474	0·8136	0·7175	0·6426	0·5816	0·5306	0·4872	0·4498	0·4173	0·3034
1·2	1·0967	0·9948	0·8587	0·7544	0·6736	0·6086	0·5546	0·5088	0·4694	0·4352	0·4052	0·2982
1·4	0·8471	0·8163	0·7512	0·6832	0·6225	0·5698	0·5242	0·4845	0·4497	0·4190	0·3917	0·2922
1·6	0·7047	0·6910	0·6566	0·6138	0·5704	0·5296	0·4924	0·4589	0·4288	0·4017	0·3773	0·2857
1·8	0·6080	0·6005	0·5803	0·5525	0·5216	0·4906	0·4609	0·4331	0·4075	0·3839	0·3623	0·2788
2·0	0·5366	0·5320	0·5190	0·5002	0·4780	0·4544	0·4309	0·4081	0·3864	0·3661	0·3472	0·2715
3·0	0·3432	0·3423	0·3394	0·3349	0·3290	0·3218	0·3138	0·3051	0·2960	0·2868	0·2775	0·2340

Table 6. Values of the integral $a\int_0^\infty J_1(at)\,J_0(bt)\,\mathrm{e}^{-ct}\,\mathrm{d}t$ for ranges of values of $\rho=b/a$ and $\zeta=c/a$

ρ \ ζ	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0	3·0
0·0	1·0000	0·8039	0·6286	0·4855	0·3753	0·2929	0·2318	0·1863	0·1520	0·1258	0·1056	0·0513
0·2	1·0000	0·7983	0·6201	0·4771	0·3683	0·2876	0·2279	0·1835	0·1500	0·1244	0·1045	0·0510
0·4	1·0000	0·7789	0·5924	0·4508	0·3473	0·2720	0·2167	0·1754	0·1442	0·1202	0·1014	0·0502
0·6	1·0000	0·7349	0·5377	0·4043	0·3124	0·2470	0·1989	0·1628	0·1351	0·1135	0·0965	0·0488
0·8	1·0000	0·6301	0·4433	0·3368	0·2658	0·2147	0·1762	0·1465	0·1234	0·1049	0·0901	0·0470
1·0	0·5000	0·3828	0·3105	0·2559	0·2130	0·1787	0·1510	0·1286	0·1102	0·0951	0·0827	0·0449
1·2	0·0000	0·1544	0·1871	0·1795	0·1621	0·1433	0·1257	0·1103	0·0965	0·0848	0·0748	0·0424
1·4	0·0000	0·0717	0·1101	0·1216	0·1197	0·1120	0·1024	0·0925	0·0831	0·0744	0·0667	0·0398
1·6	0·0000	0·0400	0·0682	0·0829	0·0876	0·0865	0·0824	0·0768	0·0707	0·0646	0·0589	0·0370
1·8	0·0000	0·0249	0·0449	0·0580	0·0647	0·0668	0·0659	0·0633	0·0597	0·0557	0·0516	0·0343
2·0	0·0000	0·0168	0·0312	0·0418	0·0485	0·0519	0·0528	0·0520	0·0502	0·0477	0·0450	0·0315
3·0	0·0000	0·0042	0·0082	0·0118	0·0149	0·0174	0·0193	0·0207	0·0216	0·0221	0·0222	0·0198

Table 7. Values of the integral $a\int_0^\infty J_0(at)\,J_1(bt)\,\mathrm{e}^{-ct}\,\mathrm{d}t$ for ranges of values of $\rho=b/a$ and $\zeta=c/a$

ρ \ ζ	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0	3·0
0·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
0·2	0·0000	0·0196	0·0330	0·0385	0·0385	0·0355	0·0315	0·0274	0·0238	0·0206	0·0178	0·0095
0·4	0·0000	0·0448	0·0723	0·0813	0·0789	0·0716	0·0630	0·0546	0·0471	0·0407	0·0353	0·0188
0·6	0·0000	0·0863	0·1267	0·1322	0·1227	0·1085	0·0940	0·0810	0·0697	0·0602	0·0522	0·0279
0·8	0·0000	0·1761	0·2073	0·1927	0·1687	0·1450	0·1239	0·1061	0·0909	0·0785	0·0681	0·0366
1·0	0·5000	0·3828	0·3105	0·2559	0·2130	0·1787	0·1510	0·1286	0·1102	0·0951	0·0827	0·0449
1·2	0·8333	0·5389	0·3911	0·3065	0·2492	0·2068	0·1740	0·1478	0·1270	0·1099	0·0957	0·0526
1·4	0·7143	0·5465	0·4210	0·3346	0·2731	0·2273	0·1918	0·1637	0·1410	0·1224	0·1071	0·0597
1·6	0·6250	0·5137	0·4182	0·3430	0·2850	0·2399	0·2042	0·1755	0·1520	0·1327	0·1166	0·0661
1·8	0·5556	0·4750	0·4016	0·3392	0·2878	0·2459	0·2117	0·1835	0·1602	0·1407	0·1243	0·0719
2·0	0·5000	0·4384	0·3806	0·3291	0·2846	0·2469	0·2151	0·1884	0·1658	0·1466	0·1303	0·0769
3·0	0·3333	0·3091	0·2853	0·2624	0·2407	0·2203	0·2014	0·1840	0·1682	0·1539	0·1409	0·0928

Table 8. Values of the integral $a\int_0^\infty J_1(at)\,J_1(bt)\,\mathrm{e}^{-ct}\,\mathrm{d}t$ for ranges of values of $\rho=b/a$ and $\zeta=c/a$

ρ \ ζ	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0	3·0
0·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
0·2	0·1015	0·0954	0·0804	0·0628	0·0472	0·0350	0·0259	0·0194	0·0147	0·0113	0·0089	0·0032
0·4	0·2134	0·1979	0·1622	0·1238	0·0917	0·0675	0·0500	0·0375	0·0285	0·0220	0·0173	0·0062
0·6	0·3530	0·3163	0·2443	0·1789	0·1298	0·0949	0·0703	0·0529	0·0405	0·0314	0·0248	0·0090
0·8	0·5721	0·4573	0·3151	0·2197	0·1570	0·1147	0·0854	0·0648	0·0500	0·0391	0·0311	0·0117
1·0	∞	0·5456	0·3422	0·2355	0·1693	0·1252	0·0945	0·0726	0·0567	0·0448	0·0359	0·0139
1·2	0·5235	0·4278	0·3072	0·2237	0·1666	0·1265	0·0976	0·0764	0·0605	0·0485	0·0393	0·0158
1·4	0·3293	0·3026	0·2482	0·1958	0·1535	0·1208	0·0958	0·0766	0·0619	0·0504	0·0414	0·0174
1·6	0·2338	0·2228	0·1962	0·1650	0·1358	0·1110	0·0907	0·0743	0·0611	0·0506	0·0422	0·0185
1·8	0·1767	0·1711	0·1566	0·1377	0·1180	0·0997	0·0838	0·0703	0·0590	0·0496	0·0420	0·0194
2·0	0·1390	0·1358	0·1272	0·1152	0·1018	0·0885	0·0762	0·0653	0·0559	0·0478	0·0410	0·0199
3·0	0·0580	0·0576	0·0562	0·0541	0·0514	0·0483	0·0449	0·0414	0·0380	0·0346	0·0314	0·0190

Table 9. Values of the integral $\int_0^\infty J_0(at)\,J_1(bt)\,\mathrm{e}^{-ct}\,\frac{\mathrm{d}t}{t}$ for ranges of values of $\rho=b/a$ and $\zeta=c/a$

ρ \ ζ	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0	3·0
0·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
0·2	0·1005	0·0985	0·0931	0·0858	0·0780	0·0706	0·0639	0·0580	0·0529	0·0485	0·0446	0·0316
0·4	0·2043	0·1996	0·1875	0·1719	0·1558	0·1407	0·1272	0·1155	0·1054	0·0965	0·0889	0·0630
0·6	0·3157	0·3064	0·2845	0·2582	0·2326	0·2094	0·1892	0·1717	0·1567	0·1437	0·1325	0·0941
0·8	0·4441	0·4234	0·3837	0·3434	0·3072	0·2759	0·2490	0·2258	0·2065	0·1896	0·1749	0·1248
1·0	0·6366	0·5499	0·4810	0·4246	0·3778	0·3388	0·3059	0·2781	0·2542	0·2338	0·2160	0·1549
1·2	0·7926	0·6582	0·5668	0·4977	0·4424	0·3970	0·3591	0·3273	0·2996	0·2759	0·2554	0·1842
1·4	0·8566	0·7310	0·6350	0·5600	0·4995	0·4497	0·4079	0·3725	0·3421	0·3158	0·2929	0·2127
1·6	0·8937	0·7800	0·6871	0·6113	0·5488	0·4965	0·4522	0·4143	0·3816	0·3532	0·3283	0·2403
1·8	0·9177	0·8147	0·7272	0·6533	0·5908	0·5375	0·4919	0·4525	0·4182	0·3881	0·3617	0·2669
2·0	0·9342	0·8404	0·7586	0·6878	0·6265	0·5735	0·5274	0·4871	0·4517	0·4205	0·3929	0·2925
3·0	0·9716	0·9074	0·8479	0·7932	0·7429	0·6968	0·6547	0·6162	0·5810	0·5488	0·5193	0·4045

Table 10. Values of the integral $\int_0^\infty J_1(at)\,J_0(bt)\,e^{-ct}\frac{dt}{t}$ for ranges of values of $\rho=b/a$ and $\zeta=c/a$

$\rho \backslash \zeta$	**0·0**	**0·2**	**0·4**	**0·6**	**0·8**	**1·0**	**1·2**	**1·4**	**1·6**	**1·8**	**2·0**	**3·0**
0·0	1·0000	0·8198	0·6770	0·5662	0·4806	0·4142	0·3621	0·3205	0·2868	0·2591	0·2361	0·1623
0·2	0·9899	0·8103	0·6690	0·5599	0·4759	0·4107	0·3594	0·3185	0·2853	0·2580	0·2352	0·1620
0·4	0·9587	0·7812	0·6448	0·5412	0·4619	0·4004	0·3518	0·3128	0·2809	0·2546	0·2326	0·1610
0·6	0·9028	0·7300	0·6041	0·5108	0·4396	0·3841	0·3397	0·3037	0·2740	0·2493	0·2283	0·1595
0·8	0·8125	0·6531	0·5478	0·4706	0·4107	0·3630	0·3240	0·2921	0·2650	0·2422	0·2227	0·1574
1·0	0·6366	0·5499	0·4810	0·4246	0·3778	0·3388	0·3059	0·2781	0·2542	0·2338	0·2160	0·1549
1·2	0·4685	0·4506	0·4152	0·3783	0·3440	0·3135	0·2866	0·2628	0·2425	0·2244	0·2084	0·1519
1·4	0·3860	0·3784	0·3597	0·3362	0·3119	0·2887	0·2672	0·2477	0·2302	0·2145	0·2004	0·1486
1·6	0·3306	0·3265	0·3154	0·3001	0·2829	0·2655	0·2485	0·2326	0·2179	0·2043	0·1920	0·1449
1·8	0·2900	0·2874	0·2803	0·2699	0·2576	0·2444	0·2311	0·2181	0·2058	0·1943	0·1836	0·1411
2·0	0·2587	0·2569	0·2521	0·2447	0·2356	0·2256	0·2151	0·2046	0·1943	0·1846	0·1753	0·1372
3·0	0·1691	0·1686	0·1674	0·1654	0·1627	0·1595	0·1558	0·1518	0·1476	0·1432	0·1387	0·1174

Table 11. Values of the integral $\int_0^\infty J_1(at)\,J_1(bt)\,e^{-ct}\frac{dt}{t}$ for ranges of values of $\rho=b/a$, and $\zeta=c/a$.

$\rho \backslash \zeta$	**0·0**	**0·2**	**0·4**	**0·6**	**0·8**	**1·0**	**1·2**	**1·4**	**1·6**	**1·8**	**2·0**	**3·0**
0·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
0·2	0·1000	0·0801	0·0624	0·0481	0·0372	0·0290	0·0230	0·0185	0·0151	0·0125	0·0105	0·0051
0·4	0·2000	0·1584	0·1222	0·0937	0·0723	0·0565	0·0448	0·0362	0·0296	0·0246	0·0207	0·0101
0·6	0·3000	0·2320	0·1758	0·1337	0·1031	0·0809	0·0645	0·0523	0·0430	0·0359	0·0303	0·0150
0·8	0·4000	0·2944	0·2180	0·1651	0·1279	0·1010	0·0812	0·0663	0·0549	0·0460	0·0390	0·0196
1·0	0·5000	0·3282	0·2421	0·1853	0·1453	0·1161	0·0943	0·0777	0·0649	0·0548	0·0468	0·0240
1·2	0·4167	0·3193	0·2464	0·1939	0·1552	0·1261	0·1039	0·0866	0·0730	0·0621	0·0534	0·0280
1·4	0·3571	0·2931	0·2379	0·1937	0·1589	0·1317	0·1101	0·0930	0·0792	0·0680	0·0589	0·0316
1·6	0·3125	0·2665	0·2244	0·1883	0·1583	0·1337	0·1136	0·0971	0·0836	0·0725	0·0633	0·0349
1·8	0·2778	0·2428	0·2099	0·1805	0·1549	0·1332	0·1149	0·0995	0·0866	0·0758	0·0667	0·0377
2·0	0·2500	0·2224	0·1960	0·1718	0·1501	0·1310	0·1146	0·1005	0·0884	0·0780	0·0691	0·0402
3·0	0·1667	0·1551	0·1437	0·1326	0·1221	0·1121	0·1028	0·0941	0·0857	0·0789	0·0723	0·0476

Table 1*a*. Variation of $\frac{a}{c}\{(a-b)^2+c^2\}\int_0^\infty J_0(at)\,J_0(bt)\,e^{-ct}\,t\,dt$ with $\rho=b/a$ and $\zeta=c/a$

$\rho \backslash \zeta$	**0·0**	**0·2**	**0·4**	**0·6**	**0·8**
0·4	0·5308	0·5305	0·5282	0·5215	0·5093
0·6	0·4267	0·4285	0·4319	0·4335	0·4310
0·8	0·3604	0·3634	0·3690	0·3737	0·3754
1·0	0·3183	0·3218	0·3274	0·3324	0·3353
1·2	0·2933	0·2952	0·2991	0·3030	0·3056
1·4	0·2752	0·2762	0·2785	0·2811	0·2830
1·6	0·2604	0·2610	0·2622	0·2638	0·2650

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TABLE 2a. VARIATION OF $\{(a-b)^2+c^2\}\int_0^\infty J_0(at)\,J_1(bt)\,\mathrm{e}^{-ct}\,t\,\mathrm{d}t$ WITH $\rho = b/a$ AND $\zeta = c/a$

$\rho \backslash \zeta$	0·0	0·2	0·4	0·6	0·8
0·4	−0·0875	−0·0755	−0·0450	−0·0072	0·0283
0·6	−0·0789	−0·0650	−0·0304	0·0117	0·0519
0·8	−0·0493	−0·0336	0·0022	0·0439	0·0837
1·0	0·0000	0·0170	0·0496	0·0866	0·1223
1·2	0·0721	0·0824	0·1065	0·1358	0·1654
1·4	0·1428	0·1491	0·1654	0·1871	0·2102
1·6	0·2062	0·2103	0·2214	0·2370	0·2544

TABLE 3a. VARIATION OF $\{(a-b)^2+c^2\}\int_0^\infty J_1(at)\,J_0(bt)\,\mathrm{e}^{-ct}\,t\,\mathrm{d}t$ WITH $\rho = b/a$ AND $\zeta = c/a$

$\rho \backslash \zeta$	0·0	0·2	0·4	0·6	0·8
0·4	0·4109	0·4154	0·4257	0·4352	0·4391
0·6	0·2257	0·2352	0·2582	0·2845	0·3070
0·8	0·0903	0·1039	0·1346	0·1697	0·2021
1·0	0·0000	0·0170	0·0496	0·0866	0·1223
1·2	−0·0426	−0·0311	−0·0039	0·0297	0·0640
1·4	−0·0643	−0·0564	−0·0358	−0·0079	0·0226
1·6	−0·0763	−0·0706	−0·0548	−0·0323	−0·0063

TABLE 4a. VARIATION OF $\frac{a}{c}\{(a-b)^2+c^2\}\int_0^\infty J_1(at)\,J_1(bt)\,\mathrm{e}^{-ct}\,t\,\mathrm{d}t$ WITH $\rho = b/a$ AND $\zeta = c/a$

$\rho \backslash \zeta$	0·0	0·2	0·4	0·6	0·8
0·4	0·2999	0·2877	0·2562	0·2159	0·1754
0·6	0·3349	0·3221	0·2895	0·2483	0·2067
0·8	0·3377	0·3243	0·2930	0·2550	0·2166
1·0	0·3183	0·3047	0·2778	0·2457	0·2130
1·2	0·2799	0·2718	0·2524	0·2277	0·2014
1·4	0·2425	0·2376	0·2245	0·2064	0·1860
1·6	0·2104	0·2072	0·1981	0·1850	0·1696

TABLE 5a. VARIATION OF $a\int_0^\infty J_0(at)\,J_0(bt)\,\mathrm{e}^{-ct}\,\mathrm{d}t+\frac{1}{2\pi}\ln\left\{\frac{(b-a)^2+c^2}{64a^2}\right\}$ WITH $\rho = b/a$ AND $\zeta = c/a$

$\rho \backslash \zeta$	0·0	0·2	0·4	0·6	0·8
0·4	0·2196	0·2084	0·1806	0·1470	0·1145
0·6	0·1610	0·1488	0·1225	0·0947	0·0700
0·8	0·0960	0·0811	0·0591	0·0407	0·0257
1·0	0·0000	−0·0021	−0·0062	−0·0109	−0·0154
1·2	−0·0775	−0·0691	−0·0593	−0·0533	−0·0497
1·4	−0·1065	−0·1017	−0·0921	−0·0827	−0·0749
1·6	−0·1198	−0·1168	−0·1093	−0·1003	−0·0915

TABLE 6a†. VARIATION OF $\frac{1}{\pi}\tan^{-1}\left(\frac{c}{b-a}\right)-a\int_0^\infty J_1(at)\,J_0(bt)\,e^{-ct}\,dt$ WITH $\rho=b/a$ AND $\zeta=c/a$

ρ \ ζ	0·0	0·2	0·4	0·6	0·8
0·4	0·0000	0·1187	0·2204	0·2992	0·3575
0·6	0·0000	0·1176	0·2123	0·2829	0·3352
0·8	0·0000	0·1199	0·2043	0·2656	0·3122
1·0	0·0000	0·1172	0·1895	0·2441	0·2870
1·2	0·0000	0·0956	0·1653	0·2180	0·2599
1·4	0·0000	0·0759	0·1399	0·1912	0·2328
1·6	0·0000	0·0625	0·1190	0·1671	0·2075

†If $b<a$ set $\tan^{-1}\frac{c}{b-a}=\pi-\tan^{-1}\frac{c}{a-b}$.

TABLE 7a†. VARIATION OF $\frac{a}{b}\left\{1-\frac{1}{\pi}\tan^{-1}\left(\frac{c}{b-a}\right)\right\}-a\int_0^\infty J_0(at)\,J_1(bt)\,e^{-ct}\,dt$ WITH $\rho=b/a$ AND $\zeta=c/a$

ρ \ ζ	0·0	0·2	0·4	0·6	0·8
0·4	0·0000	0·2112	0·3956	0·5437	0·6590
0·6	0·0000	0·1597	0·2900	0·3892	0·4647
0·8	0·0000	0·1364	0·2332	0·3043	0·3588
1·0	0·0000	0·1172	0·1895	0·2441	0·2870
1·2	0·0000	0·0861	0·1485	0·1955	0·2325
1·4	0·0000	0·0624	0·1147	0·1562	0·1894
1·6	0·0000	0·0473	0·0898	0·1257	0·1555

† If $b<a$ set $\tan^{-1}\frac{c}{b-a}=\pi-\tan^{-1}\frac{c}{a-b}$.

TABLE 8a. VARIATION OF $a\int_0^\infty J_1(at)\,J_1(bt)\,e^{-ct}\,dt+\frac{2}{\pi}+\frac{1}{2\pi}\ln\left\{\frac{(b-a)^2+c^2}{64a^2}\right\}$ WITH $\rho=b/a$ AND $\zeta=c/a$

ρ \ ζ	0·0	0·2	0·4	0·6	0·8
0·4	0·0255	0·0268	0·0328	0·0462	0·0664
0·6	0·0361	0·0349	0·0377	0·0495	0·0690
0·8	0·0345	0·0301	0·0337	0·0486	0·0703
1·0	0·0000	0·0080	0·0252	0·0476	0·0729
1·2	−0·0141	0·0005	0·0258	0·0526	0·0799
1·4	0·0124	0·0212	0·0416	0·0664	0·0927
1·6	0·0459	0·0517	0·0669	0·0874	0·1106

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8. Recurrence relations

The simplest recurrence relations are obtained by making use of the well-known result

$$\frac{2\nu}{z} J_\nu(z) = J_{\nu+1}(z) + J_{\nu-1}(z),$$

which gives the recurrence relations

$$a\{I(\mu+1,\nu;\lambda) + I(\mu-1,\nu;\lambda)\} = 2\mu I(\mu,\nu;\lambda-1) \tag{8·1}$$

and
$$b\{I(\mu,\nu+1;\lambda) + I(\mu,\nu-1;\lambda)\} = 2\nu I(\mu,\nu;\lambda-1). \tag{8·2}$$

Another set of recurrence relations can be obtained by writing

$$J_{\mu+1}(at)\,J_\nu(bt) = \{t^{-\mu}J_{\mu+1}(at)\}\{t^{+\nu}J_\nu(bt)\}\,t^{\mu-\nu}$$
$$= -\frac{1}{a}\left[\frac{\mathrm{d}}{\mathrm{d}t}\{t^{-\mu}J_\mu(at)\}\right]\{t^{+\nu}J_\nu(bt)\}\,t^{\mu-\nu}.$$

On integrating by parts, we find

$$aI(\mu+1,\nu;\lambda) - bI(\mu,\nu-1;\lambda) = C_{\mu,\nu} + (\mu-\nu+\lambda)\,I(\mu,\nu;\lambda-1) - cI(\mu,\nu;\lambda), \tag{8·3}$$

where
$$C_{\mu,\nu} = \begin{cases} \dfrac{a^\mu b^\nu}{2^{\mu+\nu}\Gamma(\mu+1)\,\Gamma(\nu+1)} & \text{if} \quad \lambda+\mu+\nu = 0; \quad (8·4\,a) \\ 0 & \text{if} \quad \lambda+\mu+\nu > 0. \quad (8·4\,b) \end{cases}$$

Similarly by writing

$$J_{\mu+1}(at)\,J_\nu(bt) = \{t^{-\mu}J_{\mu+1}(at)\}\{t^{-\nu}J_\nu(bt)\}\,t^{\mu+\nu},$$
$$J_{\mu-1}(at)\,J_\nu(bt) = \{t^{\mu}J_{\mu-1}(at)\}\{t^{\nu}J_\nu(bt)\}\,t^{-\mu-\nu},$$
$$J_{\mu-1}(at)\,J_\nu(bt) = \{t^{\mu}J_{\mu-1}(at)\}\{t^{-\nu}J_\nu(bt)\}\,t^{-\mu+\nu},$$

and proceeding in the same kind of way we obtain the recurrence relations

$$aI(\mu+1,\nu;\lambda) + bI(\mu,\nu+1;\lambda) = C_{\mu,\nu} + (\mu+\nu+\lambda)\,I(\mu,\nu;\lambda-1) - cI(\mu,\nu;\lambda), \tag{8·5}$$

$$aI(\mu-1,\nu;\lambda) + bI(\mu,\nu-1;\lambda) = -C_{\mu,\nu} + (\mu+\nu-\lambda)\,I(\mu,\nu;\lambda-1) + cI(\mu,\nu;\lambda), \tag{8·6}$$

$$aI(\mu-1,\nu;\lambda) - bI(\mu,\nu+1;\lambda) = -C_{\mu,\nu} + (\mu-\nu-\lambda)\,I(\mu,\nu;\lambda-1) + cI(\mu,\nu;\lambda). \tag{8·7}$$

It is readily seen that equations (8·3) to (8·7) imply the recurrence relations (8·1) and (8·2). Integration by parts in which e^{-ct} or t^λ are integrated and the product of the remaining factors is differentiated does not lead to any new result.

An interesting formula is obtained by writing $\mu-1$ for μ in (8·3), $\nu-1$ for ν in (8·7), multiplying the new first equation by a, the second by b and adding; we then find that

$$(a^2-b^2)\,I(\mu,\nu;\lambda) = aC_{\mu-1,\nu} - bC_{\mu,\nu-1} + a(\mu-\nu+\lambda-1)\,I(\mu-1,\nu;\lambda-1)$$
$$+ b(\mu-\nu-\lambda+1)\,I(\mu,\nu-1;\lambda-1) - c\{aI(\mu-1,\nu;\lambda) - bI(\mu,\nu-1;\lambda)\}. \tag{8·8}$$

Similarly we can show that

$$(a^2-b^2)\,I(\mu,\nu;\lambda) = bC_{\mu,\nu+1} - aC_{\mu+1,\nu} + b(\mu-\nu+\lambda-1)\,I(\mu,\nu+1;\lambda-1)$$
$$+ a(\mu-\nu-\lambda+1)\,I(\mu+1,\nu;\lambda-1) - c\{bI(\mu,\nu+1;\lambda) - aI(\mu+1,\nu;\lambda)\}. \tag{8·9}$$

9. The integrals of most practical interest

We shall consider only those integrals for which the parameters μ, ν and λ are integral, and, in accordance with the usual convention, we shall denote integral values of μ, ν and λ by latin letters, say m, n and l. There is no loss of generality involved in assuming that both m and n are positive, since $J_{-n}(z) = (-1)^n J_n(z)$.

The integrals which are of most practical interest are:

$$\begin{array}{lll} I(0,0;\,1) & I(0,0;\,0) & \\ I(1,0;\,1) & I(1,0;\,0) & I(1,0;\,-1) \\ I(1,1;\,1) & I(1,1;\,0) & I(1,1;\,-1). \end{array}$$

We shall set up the relations between these integrals. Solution of (9·3), (9·7) with

$$(\mu,\nu;\,\lambda) = (0,1;\,1)$$

in (8·3) and (1, 0; 1) in (8·7) gives

$$I(0,0;\,1) = \frac{c}{a^2-b^2}\{aI(1,0;\,1)-bI(0,1;\,1)\}, \tag{9·1}$$

$$I(1,1;\,1) = \frac{c}{a^2-b^2}\{bI(1,0;\,1)-aI(0,1;\,1)\}. \tag{9·2}$$

Therefore if we have evaluated the integrals $I(0,1;\,1)$ and $I(1,0;\,1)$ we can readily deduce the integrals $I(0,0;\,1)$ and $I(1,1;\,1)$.

If we now set $(\mu,\nu;\,\lambda) = (0,0;\,1)$ in equation (8·5) and $(\mu,\nu;\,\lambda) = (1,1;\,1)$ in equation (8·6) we find that

$$I(1,1;\,0) = aI(0,1;\,1)+bI(1,0;\,1)-cI(1,1;\,1), \tag{9·3}$$

$$I(0,0;\,0) = aI(1,0;\,1)+bI(0,1;\,1)+cI(0,0;\,1). \tag{9·4}$$

Finally, if we set $(\mu,\nu;\,\lambda) = (1,1;\,0)$ in equation (8·6) and $(\mu,\nu;\,\lambda) = (1,0;\,0)$ in equation (8·7) we obtain the relations

$$I(1,1;\,-1) = \tfrac{1}{2}aI(0,1;\,0)+\tfrac{1}{2}bI(1,0;\,0)-\tfrac{1}{2}cI(1,1;\,0), \tag{9·5}$$

$$I(1,0;\,-1) = aI(0,0;\,0)-bI(1,1;\,0)-cI(1,0;\,0). \tag{9·6}$$

From these formulae it is obvious that we can calculate all the integrals listed above if we know the values of $I(1,0;\,1)$, $I(0,1;\,1)$, $I(1,0;0)$, $I(0,1;0)$. Theoretically we need only know one of the pair $I(1,0;\,l)$, $I(0,1;\,l)$, since the other can be obtained from it by a simple change of variable, but, in order that both integrals may be tabulated for exactly the same values of the parameters b/a and c/a, it is convenient to tabulate both integrals. Furthermore, the integrals $I(1,1;\,0)$, $I(0,0;\,0)$ and $I(0,0;\,1)$ have simple expressions in terms of tabulated elliptic functions and so can be computed directly. These numerical values can be used as a check on those obtained by the method outlined above.

Finally, we state how far the tables and the recurrence relations can be used to determine integrals not included in the tables. From the tabulated integrals we can obtain $I(m,n;\,l)$ for all m, n (not only 0, 1) and all $l \leqslant 1$, providing, of course, that the required integral is convergent and the loss in accuracy in using the recurrence relations on numerical values is not excessive. It is not, however, possible to deduce the value of $I(m,n;l)$ for $l>1$ from the tables alone. Analytical formulae for $l > 1$ can be obtained from the results of §4 by using the following differentiation formulae:

$$\begin{aligned} I(\mu, \nu; \lambda) &= -\frac{\partial}{\partial c} I(\mu, \nu; \lambda - 1) \\ &= a^{-\mu-1} \frac{\partial}{\partial a} a^{\mu+1} I(\mu+1, \nu; \lambda-1) \\ &= -a^{\mu-1} \frac{\partial}{\partial a} a^{-\mu+1} I(\mu-1, \nu; \lambda-1). \end{aligned}$$

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References

Bateman, H. 1944 *Partial differential equations of mathematical physics*. New York: Dover Publications.
Heuman, C. 1941 *J. Math. Phys.* **20**, 127, 336.
Jahnke, E. & Emde, F. 1945 *Tables of functions with formulae and curves*. New York: Dover Publications.
Jeffreys, H. & B. S. 1946 *Methods of mathematical physics*. Cambridge University Press.
Magnus, W. & Oberhettinger, F. 1949 *Special functions of mathematical physics*. New York: Chelsea Publishing Co.
Nomura, Y. 1940*a* *Sci. Rep. Tohoku Univ.* (1), **28**, 304.
Nomura, Y. 1940*b* *Sci. Rep. Tohoku Univ.* (1), **29**, 22.
Nomura, Y. 1941 *Proc. Phys.-Math. Soc. Japan* (3), **23**, 168.
Smythe, W. R. 1939 *Static and dynamic electricity*. New York: McGraw-Hill Publishing Co.
Sneddon, I. N. 1946 *Proc. Camb. Phil. Soc.* **42**, 260.
Sneddon, I. N. 1951 *Fourier transforms*. New York: McGraw-Hill Publishing Co.
Sura-Bura, M. R. 1950 *Doklady Akad. Nauk S.S.S.R* (n.s.), **73**, 901.
Tallqvist, H. 1932 *Comment. Phys.-Math. Soc. Sci. Fenn*, **6**, part 11.
Terazawa, K. 1916 *J. Coll. Sci. Imp. Univ. Tokyo*, **37**, article 7.
Watson, G. N. 1944 *The theory of Bessel functions*, 2nd ed. Cambridge University Press.

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